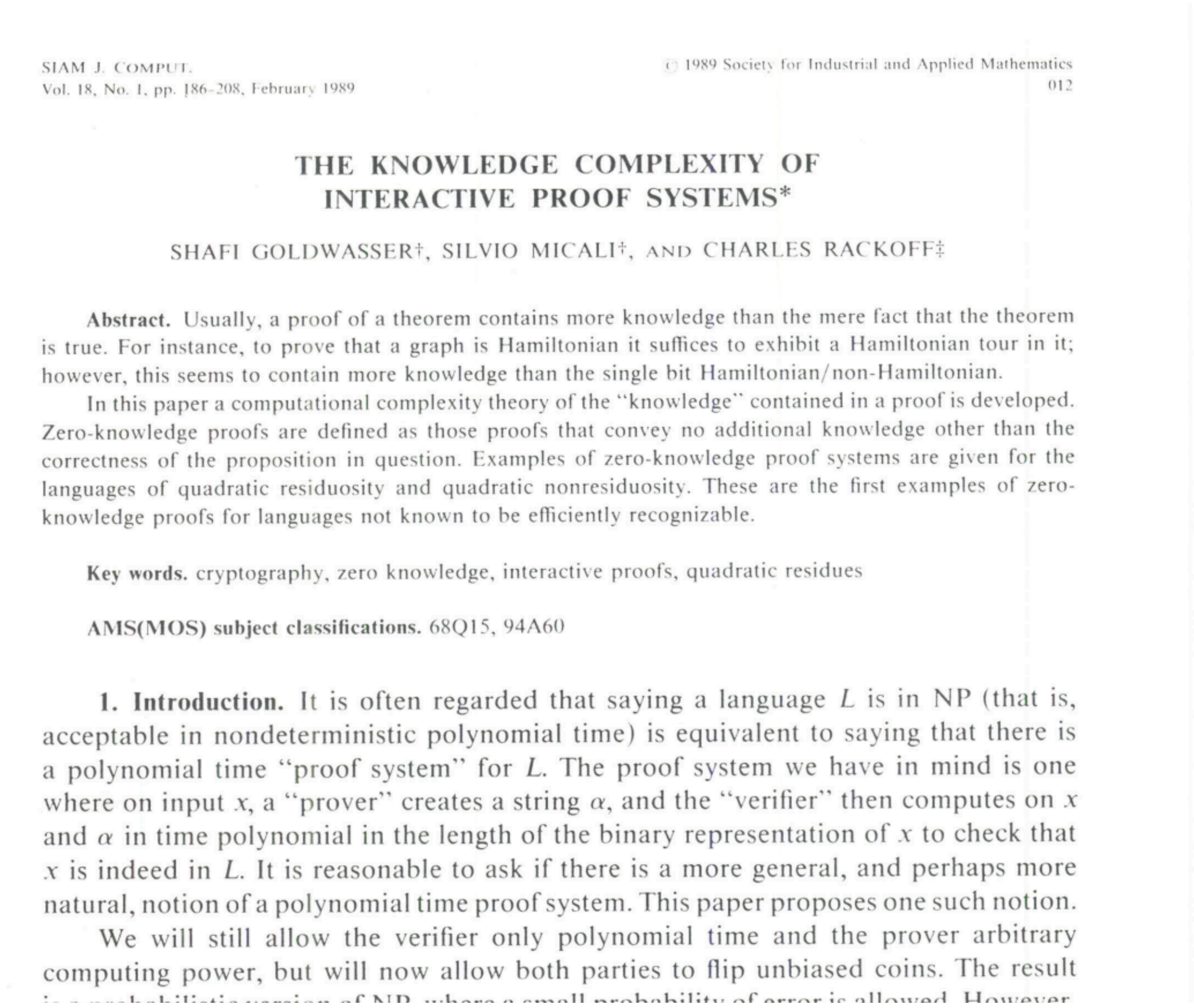
Zero-Knowledge Proofs of Knowledge

Material taken from here, here, here

Redefining classical proofs

- Proof redefined as a game between a prover and a verifier
- Game can be **interactive**, where the verifier asks questions and the prover answers
- Further generalization to a **probabilistic** proof system
- **Zero-knowledge**: “Prove that I could prove it if I felt like it”



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THE KNOWLEDGE COMPLEXITY OF
INTERACTIVE PROOF SYSTEMS*

SHAFI GOLDWASSER†, SILVIO MICALI†, AND CHARLES RACKOFF‡

Abstract. Usually, a proof of a theorem contains more knowledge than the mere fact that the theorem is true. For instance, to prove that a graph is Hamiltonian it suffices to exhibit a Hamiltonian tour in it; however, this seems to contain more knowledge than the single bit Hamiltonian/non-Hamiltonian.

In this paper a computational complexity theory of the “knowledge” contained in a proof is developed. Zero-knowledge proofs are defined as those proofs that convey no additional knowledge other than the correctness of the proposition in question. Examples of zero-knowledge proof systems are given for the languages of quadratic residuosity and quadratic nonresiduosity. These are the first examples of zero-knowledge proofs for languages not known to be efficiently recognizable.

Key words. cryptography, zero knowledge, interactive proofs, quadratic residues

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1. Introduction. It is often regarded that saying a language L is in NP (that is, acceptable in nondeterministic polynomial time) is equivalent to saying that there is a polynomial time “proof system” for L. The proof system we have in mind is one where on input x, a “prover” creates a string α, and the “verifier” then computes on x and α in time polynomial in the length of the binary representation of x to check that x is indeed in L. It is reasonable to ask if there is a more general, and perhaps more natural, notion of a polynomial time proof system. This paper proposes one such notion.

We will still allow the verifier only polynomial time and the prover arbitrary computing power, but will now allow both parties to flip unbiased coins. The result

Proofs of knowledge

- In a regular ZK proof, the prover attempts to convince the verifier that some fact is true
  - "X is true"
- In a proof of knowledge, the prover attempts convince the verifier that it knows some secret information
  - "I know why X is true"
- **Definition:** An interactive proof system (P, V) is a proof of knowledge for an NP relation R if there exists an efficient extractor E such that for any x and any prover P':
$Pr[w \leftarrow E(x) : (x, w) \in R] \geq Pr[\langle P', V \rangle(x) = 1] - \epsilon$
  - w is the witness, ϵ is the knowledge error; soundness error of at most ϵ

Schnorr Protocol

Prover wants to prove that it knows the discrete logarithm x of some group element $h = g^x \in \mathbb{G}$

$$r \leftarrow \mathbb{Z}_q, u = g^r$$

$$x, h = g^x \qquad \xrightarrow{\quad u \quad} \qquad h = g^x$$

$$\xleftarrow{\quad c \quad} \qquad c \leftarrow \mathbb{Z}_q$$

$$z \leftarrow r + cx \qquad \xrightarrow{\quad z \quad} \qquad g^z = u \cdot h^c$$

Schnorr protocol

- **Completeness**: if $z = r + cx$, then $g^z = g^{r+cx} = g^r \cdot (g^x)^c = u \cdot h^c$
- **Proof of knowledge**: Let P' be a possibly malicious prover that convinces the honest verifier with probability $\delta = 1$. Construct the extractor E as follows
  - Run P' to obtain an initial message u
  - Send random challenge c_1 to P' to get response z_1
  - Rewind the prover to its state after the first message
  - Send it another random challenge c_2 to get response z_2
  - Compute $x = \dfrac{z_1 - z_2}{c_1 - c_2}$

Schnorr protocol

- **Proof of knowledge (cont'd)**:
  - Since the prover succeeds with probability 1, we know that $g^{z_1} = u \cdot h^{c_1}$, and $g^{z_2} = u \cdot h^{c_2}$.
  - Therefore, $g^{z_1 - z_2} = h^{c_1 - c_2}, h = g^{\frac{z_1 - z_2}{c_1 - c_2}}, x = \dfrac{z_1 - z_2}{c_1 - c_2}$
  - Extraction fails if $c_1 = c_2$, which happens with probability $\dfrac{1}{q}$, which is also equal to the knowledge error

Schnorr protocol

- **Zero-knowledge:** let's try to construct a simulator.
    - Simulator sends $u = g^r$, verifier responds with challenge c
    - Rewind and sample $s \in \mathbb{Z}_q$, and compute $u = g^s/h^c$
    - Restart the verifier and get challenge
- Problem: a malicious verifier could respond with a different challenge c that depends on the u that it receives!

Schnorr protocol

- **Honest verifier zero-knowledge:**
  - Simulator sends $u = g^r$, verifier responds with challenge c
  - Rewind and sample $s \in \mathbb{Z}_q$, and compute $u = g^s / h^c$
  - Restart the verifier and get challenge c (verifier is honest, so it uses its random tape instead of adaptively choosing the challenge)
  - Simulator successfully answers with s, verifier checks that $g^s = g^s / h^c \cdot h^c$

Sigma protocols

- More general view of Schnorr's protocol
- Protocols of the form
  - Prover sends a first message u called a commitment
  - Verifier sends a uniformly random challenge c from a finite challenge space
  - Prover generates and sends a response z
- HVZK can be turned into full ZK
- Fiat-Shamir heuristic to transform into NIZK in the random oracle model

Today’s reading: Zerocash

Next time

- Moral character of cryptographic work
  - No paper review, just one discussion question
- Project presentations
  - Will send out peer grading forms